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VARIATIONAL FORMULATION IN THE GEOMETRICALLY NONLINEAR THIN ELASTIC SHELL THEORY†

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Abstract—The variational formulation for thin elastic shells undergoing large deflections is discussed. No restrictions are imposed on the magnitude of the rotational part of the deformation gradient, yet it is assumed that the strains remain small everywhere in the shell. Ten functionals and associated variational principles are derived, including the Hu–Washizu, the Hellinger–Reissner, the generalized complementary energy and the stationary potential energy principles. The total Lagrangian description (TLD) is exclusively used.

1. INTRODUCTION

Within the theory of shells the variational approach was applied successfully first to its simplest linear version. The literature concerning this topic is large, and Alumyäe, Aynola and Reissner, it seems, should be regarded as the earliest contributors to that field. For references, see the books by Washizu[1], Abovski *et al.*[2] and Rozin[3], which contains an exhaustive treatment of this and other related problems.

Since then, the variational setting for the shell theory was gradually extended to its more complex versions like the nonlinear theory of shallow shells (cf. the early paper by Aynola[4]) and, afterwards, the simplified versions of the moderate-rotation shell theory. Recent papers[5–8] may be mentioned here. The paper by Schmidt and Pietraszkiewicz[9] offers a profound study of variational principles within the complete version of the moderate-rotation shell theory. In [9] the Hu–Washizu, the Hellinger–Reissner and many other principles were derived on the basis of the principle of virtual work.

This paper will focus on the variational formulation of the general geometrically nonlinear first-approximation theory of thin elastic shells. This means that no restrictions are imposed on the rotational part of the deformation gradient (understood in the sense of the polar decomposition), but it is assumed that all the eigenvalues of the strain tensor are small in comparison with unity everywhere in the shell domain. Our chief goal is to pursue a self-consistent variational approach to such theory, based on the Kirchhoff-type of stress measures and the two-dimensional principle of virtual work as the fundamental static postulate. This task was partially undertaken by Galimov in [10, 11], where some principles of the Hu–Washizu, Reissner and Castigliano type were derived. However, those papers neglected the problem of the potentiality of the external loads applied to the shell—a question that appears, particularly in the case of the boundary moment, to be a factor limiting the possibilities of construction of the appropriate functionals. Besides, the functionals associated with those principles, though transformed to the undeformed configuration as a whole, still contained some variables related to the deformed shell boundary (cf. [12]) and thus were incompatible with purely Lagrangian description. Variational principles equivalent to the ones presented in this paper were brought out by Schmidt in [13–16], where a whole family of functionals was derived according to a procedure developed formerly in [9] for the moderate-rotation theory and the present author's earlier results[12, 17–20]. In particular, the functionals in [13–16] were based on a potential of the boundary forces and moments discussed in [17] and corresponding to the dead-load tractions at the lateral surface of the shell. This same type of potential was employed by the author in the

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functionals published in [12, 18–20]. Since this confines the applicability of the functionals in question only to this specific load, we discuss in detail the general condition for the existence of the boundary moment potential in this paper.

A different approach was suggested by Simmonds and Danielson[21] and recently developed by Atluri[22] and Valid[23]. Their papers resort directly to the polar decomposition of the deformation gradient tensor and the functionals include pure stretches and rigid rotations together with the corresponding conjugate stress measures as the independent variable fields.

This paper extends the results presented in [12, 17–20], wherein the Hu–Washizu and subsequent variational principles were derived to a more general context. This work is particularly based on the papers [12, 20] containing the relations of the general geometrically nonlinear theory of thin elastic shells reformulated in the entirely Lagrangian description with displacements as the main independent variables. However, in accordance with the remark made earlier we admit here a broader class of moments acting at the boundary of the shell. In the following section we repeat briefly the basic relations derived in [12, 20]. Then, assuming the principle of virtual work as the starting point, a procedure is followed to yield 10 functionals and associated variational principles. These functionals depend on combinations of strain, displacement and stress fields. At first an initial functional I, subject to some subsidiary conditions, is constructed. The resulting free functionals, eight in number, are grouped in two equivalent families $\mathscr{F}$ and $\mathscr{J}$. The family $\mathscr{F}$ contains the Hu–Washizu and the Hellinger–Reissner variational principles, whereas the family $\mathscr{J}$ includes the generalized complementary energy principle among the others. The accompanying free functionals form a basis suitable for deriving further functionals, with or without subsidiary conditions, and variational principles associated with them. For this end one should employ the well-known techniques concerning variational problems in general (cf. [1–3, 9, 24–26]). To exemplify this we derive the functional of the total potential energy Π.

The reduction of the present results to the consistently simplified versions of the nonlinear thin elastic shell theory[27, 28] would lead to the appropriate families of variational principles and associated functionals, for example, it can be shown that if the rotations occurring in the shell do not exceed the moderate-rotation range the functionals presented here will reduce to those published in [9]. If, on the other hand, the rotations around the tangent to the middle surface are large, whereas those around the normal remain small, the Hu–Washizu functional derived here reduces to its simplified versions given in [28] for various degrees of accuracy of the strain energy function.

2. PRELIMINARY RELATIONS

The main purpose of this section is to outline in brief the relations governing behavior of a shell undergoing finite rotations but small strains within the range of the first-approximation theory[29–31] and to put them into the form of a nonlinear boundary-value problem with displacements as the basic independent variable. These relations were derived in detail in [12, 20].

We shall clarify at first the calculus conventions adopted here. The composition of two second-order tensors $\mathbf{S}$ and $\mathbf{T}$ will be denoted by $\mathbf{ST} = S^{ij}T_{jk}\mathbf{g}_i \otimes \mathbf{g}^k$, whereas the full contraction by $\mathbf{S}\cdot\mathbf{T} = S^{ij}T_{ij}$. By $\mathbf{1} = \mathbf{g}_i \otimes \mathbf{g}^i$ we shall denote the identity (unit) tensor. The symbols $\mathbf{T}'$, $\mathbf{T}^s$, $\mathbf{T}^a$ denote the transpose, the symmetric part and the antisymmetric (skew) part of the tensor $\mathbf{T}$, respectively. We shall also employ the following operations: cross product of a vector $\mathbf{v}$ and a second-order tensor $\mathbf{T}$, denoted $\mathbf{v}\times\mathbf{T} = v^iT^{jk}(\mathbf{g}_i\times\mathbf{g}_j)\otimes\mathbf{g}_k = -(\mathbf{T}'\times\mathbf{v})'$ with a second-order tensor as a result; cross product of two second-order tensors $\mathbf{S}\times\mathbf{T} = S^{ij}T^{kl}\mathbf{g}_i\otimes(\mathbf{g}_j\times\mathbf{g}_k)\otimes\mathbf{g}_l$ with a third-order tensor as a result. In particular for the third-order Levi–Civita permutation tensor $\mathbf{E} = e^{ijk}\mathbf{g}_i\otimes\mathbf{g}_j\otimes\mathbf{g}_k$ we have $\mathbf{E} = -\mathbf{1}\times\mathbf{1}$. Finally, we recall that every second-order antisymmetric tensor $\mathbf{T}^a$ has a unique representation in the form $\mathbf{T}^a = \mathbf{v}\times\mathbf{1} = \mathbf{1}\times\mathbf{v}$, where the axial vector $\mathbf{v}$ is equal to $\mathbf{v} = -\frac{1}{2}\mathbf{E}\cdot\mathbf{T} = -\frac{1}{2}e^{ijk}T_{jk}\mathbf{g}_i$. The following sequence of operations should be observed: the

cross product must be performed prior to the composition and full contraction. In other cases brackets will indicate the right order.

The shell $\mathscr{P}$ in the undeformed state occupies a simply connected region in $\mathscr{R}^3$. Its middle surface $\mathscr{M}$ is a regular simply connected surface delimited by a piece-wise regular boundary contour $\mathscr{C}$. The vector $\mathbf{r} = \mathbf{r}(\theta^\alpha)$, $\alpha = 1, 2$, is the position vector of the surface $\mathscr{M}$ and θ^α is a curvilinear parametrization with the corresponding natural basis $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$ (comma denotes differentiation).

In the sequel we shall consider differentiable tensor fields defined on $\mathscr{M}$ and operators acting between them. Let $\mathscr{V}_k$, $k = 1, \ldots, N$, and $\mathscr{U}$ be any finite-dimensional Euclidean vector spaces and $\mathscr{T} = \bigotimes_{k=1}^{N} \mathscr{V}_k$ be a tensor product space. Then the total differential of a possibly nonlinear function $\mathbf{T} = \mathbf{T}(\mathbf{u})$, $\mathbf{T} \in \mathscr{T}$, $\mathbf{u} \in \mathscr{U}$, is denoted by $\mathrm{d}\mathbf{T} = \mathbf{T} \otimes \nabla_u \mathrm{d}\mathbf{u}$, $\mathrm{d}\mathbf{T} \in \mathscr{T}$, $\mathrm{d}\mathbf{u} \in \mathscr{U}$, where $(\,)\nabla_u = \partial(\,)/\partial u^i \mathbf{g}^i$ is the derivative operator and $\mathrm{d}\mathbf{u} = \mathrm{d}u^i \mathbf{g}_i$. The gradient $\mathbf{T} \otimes \nabla_u = \partial \mathbf{T}/\partial u^i \otimes \mathbf{g}^i$ is a tensor in the space $\mathscr{T} \otimes \mathscr{U}^*$, where $\mathscr{U}^*$ denotes the conjugate space of $\mathscr{U}$, and the partial differentiation with respect to u^i should be applied to the product $\mathbf{T} = \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_N$ according to the Leibniz rule. If, further, $\mathscr{V}_N = \mathscr{U}^*$ and $\dim \mathscr{U}^* = 3$ we define the divergence and curl as $\mathbf{T}\nabla_u = (\partial \mathbf{T}/\partial u^i)\mathbf{g}^i$ and $\mathbf{T} \times \nabla_u = \partial \mathbf{T}/\partial u^i) \times \mathbf{g}^i$, respectively. Finally, whenever the space $\mathscr{U}$ is identical with T_M, i.e. a plane tangent to $\mathscr{M}$ at a point $M \in \mathscr{M}$, we shall omit the subscript T_M in $(\,)\nabla_{T_M}$ writing simply $(\,)\nabla$.

Now, let $\xi = A(\theta)$, $\theta \in \Theta$, $\xi \in \Xi$, be an operator acting between the Hilbert spaces Θ and Ξ of tensor fields defined on the surface $\mathscr{M}$. Then $\delta\xi = DA(\theta, \delta\xi)$ denotes its Gâteaux differential. However, further we shall abide by the traditional concepts referring to it as to the variation of the operator A. Whenever the operator DA is linear in $\delta\theta$ we shall write $\delta\xi = (\delta A/\delta\theta)\delta\theta$, where $\delta A/\delta\theta$ denotes the Gâteaux derivative. For the precise mathematical background of the notions just introduced, see, e.g. the monographs by Vainberg[32, 33]. We assume that all functions and operators we shall deal with are differentiable up to the order needed.

Let us begin by introducing the fields inherent in the surface $\mathscr{M}$. These are $\mathbf{a} = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha$—the metric tensor, $\mathbf{n}$—the unit normal vector, $\boldsymbol{\varepsilon} = \varepsilon^{\alpha\beta}\mathbf{a}_\alpha \otimes \boldsymbol{\alpha}_\beta$—the surface permutation tensor, $\mathbf{b} = -\mathbf{n} \otimes \nabla$—the curvature tensor. Besides, $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ denote the Darboux orthonormal triad along the contour $\mathscr{C}$ with $\mathbf{t}$ as the unit tangent vector. $\mathrm{d}A$ is the area element of the surface $\mathscr{M}$, whereas $\mathrm{d}s$ the length element of the contour $\mathscr{C}$. The following relations hold true on the surface $\mathscr{M}$:

$$\begin{gathered}
\mathbf{r} \otimes \nabla = \mathbf{a} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}, \qquad \boldsymbol{\varepsilon} = -\mathbf{n} \times \mathbf{a} = -\mathbf{a} \times \mathbf{n} = -\mathbf{n} \times \mathbf{1}, \\
\boldsymbol{\varepsilon}\boldsymbol{\varepsilon} = -\mathbf{a}, \qquad \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} = 2, \qquad \mathbf{n} = -\tfrac{1}{2}\mathbf{1} \times \mathbf{1} \cdot \boldsymbol{\varepsilon}, \\
\mathbf{t} = \mathbf{r}_{,s}, \qquad \boldsymbol{\nu} = \mathbf{t} \times \mathbf{n}, \\
\mathrm{d}A = \boldsymbol{\varepsilon} \cdot (\mathrm{d}\mathbf{r}_1 \otimes \mathrm{d}\mathbf{r}_2), \qquad \mathrm{d}\mathbf{r}_1, \mathrm{d}\mathbf{r}_2 \in T_M.
\end{gathered} \tag{2.1}$$

These quantities describe the geometry of the undeformed surface $\mathscr{M}$. We assume that they are known functions of the position vector $\mathbf{r}$. Therefore the undeformed configuration of the shell will serve for the reference configuration and this means adoption of the Lagrangian description.

The shell $\mathscr{P}$ subjected to the action of some external system of forces will deform to another configuration $\bar{\mathscr{P}}$ with the middle surface $\bar{\mathscr{M}}$ delimited by the contour $\bar{\mathscr{C}}$. Quantities pertinent to the surface $\bar{\mathscr{M}}$ and equivalent to the ones already introduced on $\mathscr{M}$ will be represented by the same symbols with the addition of overbars, i.e. $\bar{\mathbf{r}}$, $\bar{\mathbf{a}}$, $\bar{\mathbf{n}}$, $(\,)\bar{\nabla}$, etc. With this modification the relations (2.1) still hold true on the surface $\bar{\mathscr{M}}$. The linear transformation mapping $\mathrm{d}\mathbf{r} \in T_M$ into $\mathrm{d}\bar{\mathbf{r}} \in \bar{T}_{\bar{M}}$ is a two-point surface deformation gradient tensor

$$\boldsymbol{\Gamma} = \bar{\mathbf{r}} \otimes \nabla = \mathbf{a} + \mathbf{u} \otimes \nabla, \tag{2.2}$$

and the vector field $\mathbf{u} = \mathbf{r} - \mathbf{r}$ is the displacement of the surface $\mathscr{M}$.

The following equations link the quantities in both configurations:

$$
\begin{gathered}
J = \frac{\mathrm{d}\bar{A}}{\mathrm{d}A} = \tfrac{1}{2}\boldsymbol{\Gamma}'\bar{\boldsymbol{\varepsilon}}\boldsymbol{\Gamma}\cdot\boldsymbol{\varepsilon}, \qquad \bar{\boldsymbol{\varepsilon}} = J^{-1}\boldsymbol{\Gamma}\boldsymbol{\varepsilon}\boldsymbol{\Gamma}', \\
\bar{\mathbf{n}} = -\tfrac{1}{2}J^{-1}\mathbf{1}\times\mathbf{1}\cdot\boldsymbol{\Gamma}\boldsymbol{\varepsilon}\boldsymbol{\Gamma}', \qquad (\,)\bar{\nabla} = (\,)\bar{\nabla}\boldsymbol{\Gamma}, \\
\bar{s} = a_t\,\mathrm{d}s, \qquad a_t = |\bar{\mathbf{a}}_t| = \sqrt{1+2\gamma_{tt}} \\
\bar{\mathbf{a}}_t = \boldsymbol{\Gamma}\mathbf{t} = a_t\bar{\mathbf{t}} = \mathbf{t}+\mathbf{u}_{,s}, \\
\bar{\boldsymbol{\nu}} = \mathbf{R}\boldsymbol{\nu}, \qquad \bar{\mathbf{t}} = \mathbf{R}\mathbf{t}, \qquad \bar{\mathbf{n}} = \mathbf{R}\mathbf{n}, \\
\mathbf{R} = \bar{\mathbf{t}}\times\bar{\mathbf{n}}\otimes\boldsymbol{\nu}+\bar{\mathbf{t}}\otimes\mathbf{t}+\bar{\mathbf{n}}\otimes\mathbf{n} = \mathbf{e}\otimes\mathbf{e}+\cos\alpha(\mathbf{1}-\mathbf{e}\otimes\mathbf{e})+\sin\alpha\mathbf{e}\times\mathbf{1}, \\
\sin\alpha\mathbf{e} = \boldsymbol{\nu}\times\bar{\boldsymbol{\nu}}+\mathbf{t}\times\bar{\mathbf{t}}+\mathbf{n}\times\bar{\mathbf{n}}, \\
2\cos\alpha+1 = \boldsymbol{\nu}\cdot\bar{\boldsymbol{\nu}}+\mathbf{t}\cdot\bar{\mathbf{t}}+\mathbf{n}\cdot\bar{\mathbf{n}}.
\end{gathered}
\tag{2.3}
$$

The orthogonal tensor $\mathbf{R}$ in (2.3) describes the total rotation of the boundary element by the angle α around the axis $\mathbf{e}$ and a_t is the stretch of this element.

The relations (2.1)–(2.3) depict the geometric aspect of the theory. When substituted to the surface deformation measures (for justification of the measures assumed see [12, 20]): the surface strain tensor $\boldsymbol{\gamma}$ and the change of curvature tensor $\boldsymbol{\chi}$,

$$
\boldsymbol{\gamma} = \tfrac{1}{2}(\boldsymbol{\Gamma}'\boldsymbol{\Gamma}-\mathbf{a}), \quad \boldsymbol{\chi} = J\boldsymbol{\Gamma}'(\bar{\mathbf{n}}\otimes\nabla)+\mathbf{b}(1+\mathbf{a}\cdot\boldsymbol{\gamma}) \tag{2.4}
$$

they yield the kinematic relations

$$
\begin{gathered}
\boldsymbol{\gamma}(\mathbf{u}) = (\mathbf{u}\otimes\nabla)^{s}+\tfrac{1}{2}(\mathbf{u}\otimes\nabla)^{t}(\mathbf{u}\otimes\nabla), \\
\boldsymbol{\chi}(\mathbf{u}) = \mathbf{m}\otimes\nabla+(\mathbf{u}\otimes\nabla)^{t}(\mathbf{m}\otimes\nabla)+\mathbf{b}[1+\mathbf{a}\cdot\boldsymbol{\gamma}(\mathbf{u})],
\end{gathered}
\tag{2.5}
$$

where

$$
\mathbf{m}(\mathbf{u}) = J\bar{\mathbf{n}} = -\tfrac{1}{2}\mathbf{1}\times\mathbf{1}\cdot\boldsymbol{\Gamma}\boldsymbol{\varepsilon}\boldsymbol{\Gamma}^{t} = \tfrac{1}{2}\varepsilon^{\alpha\beta}(\mathbf{a}_{\alpha}+\mathbf{u}_{,\alpha})\times(\mathbf{a}_{\beta}+\mathbf{u}_{,\beta}). \tag{2.6}
$$

The other aspect of the problem is related to the phenomena resulting from the action of external forces on the shell. These forces depend in general on the actual configuration $\bar{\mathscr{P}}$. Let us assume that the contour $\mathscr{C}$ consists of two separate parts: $\mathscr{C}_f$—along which the force $\mathbf{B} = \mathbf{B}(\mathbf{u})$ and the moment $\mathbf{k} = \mathbf{k}(\alpha, \mathbf{e})$ per unit length are prescribed known functions of the displacement $\mathbf{u}$, the angle of the total rotation α and the axis $\mathbf{e}$, respectively; and its completion $\mathscr{C}_u$ with a fixed configuration of the shell boundary. Similar division can be made on the middle surface $\mathscr{M}$. However, we neglect in further considerations the case wherein some parts of $\mathscr{M}$ have prescribed geometry, because it leads to problems with multiply-connected regions and seems of little practical importance. Thus we assume that the whole surface $\mathscr{M}$ is subject to the load $\mathbf{p} = \mathbf{p}(\mathbf{u})$ per unit area. Now we can impose the equilibrium condition in the global form, demanding the principle of virtual work to hold true

$$
\begin{gathered}
\iint_{\mathscr{M}}(\mathbf{N}\cdot\delta\boldsymbol{\gamma}+\mathbf{M}\cdot\delta\boldsymbol{\chi})\,\mathrm{d}A = \iint_{\mathscr{M}}\mathbf{p}\cdot\delta\mathbf{u}\,\mathrm{d}A+\int_{\mathscr{C}_f}(\mathbf{B}\cdot\delta\mathbf{u}+\mathbf{k}\cdot\delta\boldsymbol{\omega})\,\mathrm{d}s, \\
\mathbf{p} = \mathscr{F}\bar{\mathbf{p}}, \quad \mathbf{B} = a_t\bar{\mathbf{B}}, \quad \mathbf{k} = a_t\bar{\mathbf{k}}.
\end{gathered}
\tag{2.7}
$$

Here $\mathbf{N}$ and $\mathbf{M}$ denote the Lagrangian internal force and moment tensors—the counterparts of the Kirchhoff (or the second Piola–Kirchhoff) stress tensor in the three-dimensional

theory of deformable continuum[34]. For an alternative approach, based on the Jaumann-type of internal forces, see [21–23, 34].

Note the occurrence of the term $\delta\omega = \delta\omega\mathbf{i}$ in (2.7). It represents a small incremental rotation of the boundary element in the actual state by an infinitesimal angle $\delta\omega$ around the axis $\mathbf{i}$. Note also that in general $\delta\omega$ does not coincide with $\delta(\alpha\mathbf{e})$. The equations linking these terms and the boundary configuration variables can be derived from the condition for the spin tensor $\delta\omega \times \mathbf{1} = \mathbf{R}'\delta\mathbf{R}$[17] and eqns $(2.3)_{6-8}$. We have

$$\delta\omega = \frac{1}{a_t}\mathbf{t}\times\mathbf{R}'\delta\bar{\mathbf{a}}_t + \mathbf{t}(\mathbf{t}\cdot\delta\omega) = \frac{\sin\alpha}{\alpha}\mathbf{1} + \frac{\alpha-\sin\alpha}{\alpha}\mathbf{e}\otimes\mathbf{e} + \frac{1-\cos\alpha}{\alpha}\mathbf{e}\times\mathbf{1}\,\delta(\alpha\mathbf{e}), \tag{2.8}$$

$$\mathbf{t}\cdot\delta\omega = \bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}, \qquad \delta\bar{\mathbf{a}}_t = \delta\mathbf{u}_{,s}.$$

Besides, from the identities $\bar{\mathbf{n}}^2 = 1$ and $\bar{\mathbf{n}}\cdot\bar{\mathbf{a}}_t = 0$ and $(2.8)_3$ we can derive the following expression for $\delta\bar{\mathbf{n}}$ at the boundary

$$\delta\bar{\mathbf{n}} = (\mathbf{t}\cdot\delta\omega)\bar{\mathbf{v}} - \frac{1}{a_t}\bar{\mathbf{t}}(\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s}). \tag{2.9}$$

Having substituted (2.5), $(2.8)_1$ and (2.9) to (2.7) and applying the Stokes theorem, the principle of virtual work can be transformed to the equivalent form

$$-\iint_{\mathscr{M}}(\mathbf{T}\nabla+\mathbf{p})\cdot\delta\mathbf{u}\,dA + \int_{\mathscr{C}_f}\{[\mathbf{T}\nu-\mathbf{B}+[(\mathbf{F}+\mathbf{t}\times\mathbf{k})\mathbf{R}']_{,s}]\cdot\delta\mathbf{u} + (J\bar{\nu}\Gamma\mathbf{M}\nu+\mathbf{k}\cdot\mathbf{t})\mathbf{t}\cdot\delta\omega\}\,ds - \sum_{i=1}^{N}(\mathbf{F}_i+\mathbf{t}_i\times\mathbf{k}_i)\mathbf{R}'\cdot\delta\mathbf{u}_i = 0 \tag{2.10}$$

and the symbols used in (2.10) represent

$$\mathbf{T} = \Gamma[\mathbf{N}+(\mathbf{M}\cdot\mathbf{b})\mathbf{a}] + (\mathbf{m}\otimes\nabla)\mathbf{M} - [(\Gamma\mathbf{M})\nabla]\times\Gamma\varepsilon$$

$$\mathbf{F} = \left(\frac{1}{a_t}J\bar{\mathbf{t}}\Gamma\mathbf{M}\nu\right)\mathbf{n} \tag{2.11}$$

$$\mathbf{F}_i = \mathbf{F}(s-0)-\mathbf{F}(s+0).$$

The condition (2.10) yields directly the local form of the Lagrangian equilibrium conditions:

the equilibrium equations on $\mathscr{M}$

$$\mathbf{T}\nabla+\mathbf{p} = \mathbf{0}, \tag{2.12}$$

the natural boundary conditions along $\mathscr{C}_f$

$$\begin{aligned}\mathbf{T}\nu-\mathbf{B}+[(\mathbf{F}+\mathbf{t}\times\mathbf{k})\mathbf{R}']_{,s} &= \mathbf{0},\\ J\bar{\nu}\Gamma\mathbf{M}\nu+\mathbf{k}\cdot\mathbf{t} &= 0,\end{aligned} \tag{2.13}$$

and the jump conditions

$$\mathbf{F}_i+\mathbf{t}_i\times\mathbf{k}_i = \mathbf{0} \tag{2.14}$$

at the points $M_i\in\mathscr{C}_f$, $i = 1,\ldots,N$, at which either the moment $\mathbf{k}$ is discontinuous or the contour $\mathscr{C}_f$ has corners.

It remains to impose the geometric boundary conditions. This consists in prescribing the position of the deformed boundary contour $\bar{\mathscr{C}}_u$ and the orientation of its Darboux

orthonormal triad, $\bar{\nu}^*$, $\bar{\mathbf{t}}^*$, $\bar{\mathbf{n}}^*$. However, in virtue of eqns $(2.3)_{3-6}$, it suffices to specify merely the orientation of the unit normal $\bar{\mathbf{n}}^*$. Yet $\bar{\mathbf{n}}$ itself must still satisfy the kinematic constraint $\bar{\mathbf{n}} = \bar{\mathbf{t}} \times (\bar{\mathbf{n}} \times \bar{\mathbf{t}})$. Consequently, $\bar{\mathbf{n}}$ is a function of $\bar{\mathbf{t}}$ and some scalar parameter ϕ, i.e. $\bar{\mathbf{n}} = f(\bar{\mathbf{t}}, \phi)$ and, thus, so is the rotation tensor $\mathbf{R} = \mathbf{R}(\bar{\mathbf{t}}, \phi)$. This remark also concerns the static boundary conditions (2.13), where $\mathbf{R}$ appears explicitly and implicitly in the function $\mathbf{k} = \mathbf{k}(\alpha, \mathbf{e})$. However, it is not clear up to now whether there exists any standard form of such a function. In [12, 17–20], where the parameter ϕ was assumed to be $n_\nu = \bar{\mathbf{n}} \cdot \nu$, the resulting expression for $\bar{\mathbf{n}}(\bar{\mathbf{t}}, n_\nu)$ turned out to be nonunique. The same happens when ϕ is identified with the angle α of the total boundary rotation. To avoid these difficulties we impose the conditions valid for any parameter used:

$$\begin{aligned} &\mathbf{u} - \mathbf{u}^*(s) = \mathbf{0}, \\ &\bar{\mathbf{n}} \cdot \bar{\mathbf{t}}^* \times \mathbf{n}^*(\bar{\mathbf{t}}^*, \phi^*) = 0 \qquad \text{along } \mathscr{C}_u, \\ &\mathbf{u}_k - \mathbf{u}_k^*(s_k) = 0, \qquad k = 1, \ldots, K, \text{ at } M_k \in \mathscr{C}_u. \end{aligned} \tag{2.15}$$

In $(2.15)_2$ the expression $\bar{\mathbf{t}}^* = (1/a_t^*)(\mathbf{t} + \mathbf{u}_{,s}^*)$ should be substituted for $\bar{\mathbf{t}}^*$. The condition $(2.15)_3$ prescribes the displacements at the corners of $\mathscr{C}_u$.

The equations displayed in this section form a system valid for arbitrary deformations of the surface $\mathscr{M}$. They comprise two independent kinds of variables, namely variables of geometric and static character. To bind this system up into a complete boundary-value problem we still need the constitutive equations.

3. THE POTENTIALS AND THE INITIAL FUNCTIONAL

Apart from interpreting the principle of virtual work (2.7) as a balance condition, it can also be viewed as a condition equating to zero a Gâteaux differential

$$\delta I = \iint_{\mathscr{M}} (\mathbf{N} \cdot \delta\gamma + \mathbf{M} \cdot \delta\chi - \mathbf{p} \cdot \delta\mathbf{u}) \, \mathrm{d}A - \int_{\mathscr{C}_f} (\mathbf{B} \cdot \delta\mathbf{u} + \mathbf{k} \cdot \delta\omega) \, \mathrm{d}s \tag{3.1}$$

of some unknown functional I with the kinematic relations as constraints. The question, whether there really exists such a functional, is equivalent to the problem of finding potentials for the internal forces $\mathbf{N}$, $\mathbf{M}$ and the external forces $\mathbf{p}$, $\mathbf{B}$, $\mathbf{k}$ treated now as operators acting from the spaces of geometric variables to the conjugate spaces of forces. General concepts concerning the potentiality of operators are broadly discussed in [32, 33] and with specification to mechanics in [26, 34]. We only recall that if $DA(\boldsymbol{\theta}, \delta\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$, exists and is hemicontinuous in $\boldsymbol{\theta}$ and the domain Θ is convex, then the condition for potentiality demands the equality

$$\langle DA(\boldsymbol{\theta}, \delta\boldsymbol{\theta}_1), \delta\boldsymbol{\theta}_2 \rangle = \langle DA(\boldsymbol{\theta}, \delta\boldsymbol{\theta}_2), \delta\boldsymbol{\theta}_1 \rangle \tag{3.2}$$

to hold true. The corresponding functional follows from the formula

$$F(\boldsymbol{\theta}) = \int_0^1 \langle A(\tau\boldsymbol{\theta}), \boldsymbol{\theta} \rangle \, \mathrm{d}\tau. \tag{3.3}$$

Let us notice that according to (3.1) the potential of the internal forces $\mathbf{N}$, $\mathbf{M}$ (or the internal energy function) can only depend on the variables γ and χ. Formally this means that the internal energy of the shell is determined solely by the deformation of the middle surface $\mathscr{M}$. This confines applicability of the theory to those deformations, in which the principal strains remain small in comparison with unity everywhere in the shell. Otherwise (3.1) ought to include additional variables describing the deformation across the thickness h of the shall. Besides, let us consider only the elastic deformations. These restrictions coincide well with the assumptions of the so-called first-approximation theory developed

in [29–31]. Profiting directly from the results contained therein we may postulate the existence of a two-dimensional strain energy density function defined per unit area of $\mathscr{M}$ in the approximate form

$$\Sigma = \frac{h}{2}\mathbf{H}\cdot\left(\gamma\otimes\gamma+\frac{h^2}{12}\chi\otimes\chi\right), \tag{3.4}$$

where $\mathbf{H}$ denotes the fourth-order modified elasticity tensor[31]. Taking the variation of (3.4) we can obtain the constitutive equations in the form

$$\mathbf{N} = \frac{\delta\Sigma}{\delta\gamma} = h\mathbf{H}\cdot\gamma, \qquad \mathbf{M} = \frac{\delta\Sigma}{\delta\chi} = \frac{h^3}{12}\mathbf{H}\cdot\chi \tag{3.5}$$

typical for the linearly elastic materials.

As it seems impossible to characterize any universal class of potential loads $\mathbf{p}$ and $\mathbf{B}$, we assume that the corresponding potentials $\Pi(\mathbf{u})$ and $\Phi(\mathbf{u})$ exist, so that the condition (3.2) is satisfied. For the particular case of the pressure load we refer the reader to [35–37]. Yet we shall concentrate on the question of the potentiality of the moment $\mathbf{k}$. This problem was solved by Simmonds[38] for a rigid body in rotational motion in terms of the Rodrigues vector $\boldsymbol{\beta} = 2\tan(\alpha/2)\mathbf{e}$. This variable becomes singular any time $\alpha = n\pi$. We present here an alternative solution, free of this disadvantage.

Let us point out that the main difficulty here is due to the fact that the infinitesimal rotation $\delta\omega$ is not a Gâteaux differential of any vector field. To overcome this we must modify the expression for the work rate $\delta\Psi = \mathbf{k}\cdot\delta\omega$ with the aid of the relation $(2.8)_1$, which substituted for $\delta\omega$ yields

$$\delta\Psi = \mathbf{k}\cdot\delta\omega = \hat{\mathbf{k}}\cdot\delta\boldsymbol{\alpha}, \tag{3.6}$$

where the pseudo moment

$$\hat{\mathbf{k}} = \frac{\alpha-\sin\alpha}{\alpha^3}\boldsymbol{\alpha}\times(\boldsymbol{\alpha}\times\mathbf{k})+\frac{\cos\alpha-1}{\alpha^2}\boldsymbol{\alpha}\times\mathbf{k}+\mathbf{k} \tag{3.7}$$

belongs to a vector space conjugate to another space, whose members are the vectors $\boldsymbol{\alpha} = \alpha\mathbf{e}$. Further we may apply the routine methods involving (3.2), which eventually results in the condition for the potentiality of the moment in the local form

$$\hat{\mathbf{k}}\times\nabla_\alpha = \mathbf{0}. \tag{3.8}$$

Some important conclusions may be deduced from (3.8):

(i) the constancy of the moment $\mathbf{k}$, i.e. $\mathbf{k}\times\nabla_\alpha = \mathbf{0}$, does not suffice to guarantee the existence of a potential;

(ii) if $\mathbf{k} = k\mathbf{e}$ with $k = \text{const}$ the potential takes the simple form $\Psi(\alpha) = k\alpha$; moreover, if $\mathbf{e} = \mathbf{t}$, then k is the bending moment and α is the winding angle of the shell around the contour $\mathscr{C}$;

(iii) if $\mathbf{k} = \bar{\mathbf{n}}\times\mathbf{H}$ with $\mathbf{H} = \text{const}$, then a potential in the form $\Psi(\bar{\mathbf{n}}) = \mathbf{H}\cdot(\bar{\mathbf{n}}-\mathbf{n})$ exists and $\mathbf{H}$ corresponds to the resultant static moment of dead-load type tractions acting on the shell lateral boundary surface $\partial\mathscr{P}$ (see [17]).

There is still one important aspect, concerning operators and the related potentials, closely connected with the complementary variational formulations. If an operator turns out to be strictly monotone and potential, then the corresponding potential is strictly convex[33]. The converse is also true. On the other hand, strict monotonicity implies invertibility of the operator—a property enabling effectuation of the Legendre involuntary transformation[24–26]. Since the strain energy density function is strictly convex by

definition, there always exists such fourth-order tensor $\check{\mathbf{E}}$ that

$$\gamma = \frac{1}{h}\check{\mathbf{E}}\cdot\mathbf{N}, \qquad \chi = \frac{12}{h^3}\check{\mathbf{E}}\cdot\mathbf{M}. \tag{3.9}$$

From these inverted constitutive equations follows the so-called first complementary energy density function[34]

$$\Sigma^c(\mathbf{N},\mathbf{M}) = \mathbf{N}\cdot\gamma+\mathbf{M}\cdot\chi-\Sigma(\gamma,\chi) = \frac{1}{2h}\check{\mathbf{E}}\cdot\left(\mathbf{N}\otimes\mathbf{N}+\frac{12}{h^2}\mathbf{M}\otimes\mathbf{M}\right), \tag{3.10}$$

which, by virtue of the properties of the Legendre transformation (3.10), satisfies the equations

$$\gamma = \frac{\delta\Sigma^c}{\delta\mathbf{N}}, \qquad \chi = \frac{\delta\Sigma^c}{\delta\mathbf{M}}. \tag{3.11}$$

The same procedure may be applied to the potentials $\Pi(\mathbf{u})$ and $\Phi(\mathbf{u})$, provided the inverse operators $\mathbf{u} = \mathbf{u}(\mathbf{p})$ and $\mathbf{u} = \mathbf{u}(\mathbf{B})$ exist. This would result in the following relations:

$$\begin{aligned} \Pi^c(\mathbf{p}) &= \mathbf{p}\cdot\mathbf{u}-\Pi(\mathbf{u}), \qquad \delta\Pi^c = \mathbf{u}\cdot\delta\mathbf{p}, \\ \Phi^c(\mathbf{B}) &= \mathbf{B}\cdot\mathbf{u}-\Phi(\mathbf{u}), \qquad \delta\Phi^c = \mathbf{u}\cdot\delta\mathbf{B}. \end{aligned} \tag{3.12}$$

The quantities Π^c and Φ^c should be termed complementary work of the external forces $\mathbf{p}$ and $\mathbf{B}$.

Before passing to the direct examination of the functionals and variational principles, it seems worthwhile to expose a general view of the nonlinear boundary-value problem we have to deal with. A system of the following equations constitutes this problem:

(i) kinematic relations (2.5): $\varepsilon = \varepsilon(u)$;
(ii) equilibrium equations on $\mathscr{M}$ (2.11), (2.12): $\mathbf{T}(u,\sigma)\nabla+\mathbf{p} = \mathbf{0}$;
(iii) static boundary conditions along $\mathscr{C}_f$ (2.13), (2.14): $f(u_{|c}, \sigma_{|c}, \sigma^*_{|c}) = 0$;
(iv) constitutive equations (3.5): $\sigma = \sigma(\varepsilon)$, and the inverted constitutive equations (3.9): $\varepsilon = \varepsilon(\sigma)$;
(v) geometric boundary conditions along $\mathscr{C}_u$ (2.15): $g(u_{|c}, u^*_{|c}) = 0$.

Three kinds of fields occur in the equations (i)–(v): the displacements $u = \{\mathbf{u}$ on $\mathscr{M}$; $\mathbf{u}_{|c}, \alpha\mathbf{e}$ along $\mathscr{C}\}$, the strain measures $\varepsilon = \{\gamma, \chi\}$ and the conjugate stress measures $\sigma = \{\mathbf{N}, \mathbf{M}\}$. There are also parameters: fixed displacements along $\mathscr{C}_u$ $u^*_{|c} = \{\mathbf{u}^*, \alpha^*\mathbf{e}^*\}$ and prescribed functions $\mathbf{p} = \mathbf{p}(\mathbf{u})$ on $\mathscr{M}$ and $\sigma^*_{|c}(u_{|c}) = \{\mathbf{B}, \mathbf{k}\}$ along $\mathscr{C}_f$ specifying the loads with respect to the actual displacements. All conceivable fields $[u, \varepsilon, \sigma]$ form the space of states $\mathscr{S}$ of the shell $\mathscr{P}$ and the equations (i)–(v) determine these particular points $[u_0, \varepsilon_0, \sigma_0] \in \mathscr{S}$, which are the solutions to the boundary-value problem.

Resorting to the origins of this section, let us point out that the static equations (ii) and (iii) resulted from the variational principle $\delta I = 0$ after substitution of the kinematic relations (i) to (3.1). Thus, if the functional

$$I[u,\varepsilon] = \iint_{\mathscr{M}}[\Sigma(\varepsilon)-\Pi(u)]\,\mathrm{d}A - \int_{\mathscr{C}_f}[\Phi(u)+\Psi(\alpha\mathbf{e})]\,\mathrm{d}s \tag{3.13}$$

exists, the boundary-value problem (i)–(v), may be converted into an equivalent constrainted variational problem:

$$\text{find } [u_0, \varepsilon_0, \sigma_0 = \sigma(\varepsilon_0)] \in \mathscr{S} \text{ such that } I[u_0, \varepsilon_0] = \underset{[u,\varepsilon]\in\mathscr{D}}{\text{stationary values of}} I[u,\varepsilon]. \tag{3.14}$$

The subset $\mathscr{D} \subset \mathscr{S}$

$$\mathscr{D} = \{[u, \varepsilon] \,|\, \varepsilon - \varepsilon(u) = 0, g(u_{|c}, u^*) = 0\}$$

is the set of all feasible strain and displacement fields on $\mathscr{M}$.

4. GENERALIZED VARIATIONAL PRINCIPLES

The functional (3.13) must be supplemented by subsidiary conditions to acquire a definite sense. Generally, such a situation is inconvenient in practical applications, where the unconstrained (or free) formulations are preferable. Therefore in this section we shall be mainly preoccupied with transformations leading to free functionals of the Hu–Washizu type. To this end we shall profit from the well-known techniques developed by Courant and Hilbert[24] and Lanczos[25]. The books[1–3, 26] lay particular stress on the application of these ideas to mechanics. Figure 1 illustrates the procedure followed.

Let us introduce the kinematic relations (i) and the geometric boundary conditions (v) into the functional I via the Lagrange multiplier rule. This operation spreads the domain of the resulting functional I_1 on the whole space $\mathscr{S}$

$$I_1[u, \varepsilon, \sigma] = \iint_{\mathscr{M}} \{\Sigma - \Pi - \mathbf{N} \cdot [\gamma - \gamma(\mathbf{u})] - \mathbf{M} \cdot [\chi - \chi(\mathbf{u})]\} \, dA$$

$$- \int_{\mathscr{C}_f} [\Phi + \Psi] \, ds - \int_{\mathscr{C}_u} [\mathbf{P} \cdot (\mathbf{u} - \mathbf{u}^*) + M\mathbf{n} \cdot \mathbf{t}^* \times \mathbf{n}^*] \, ds - \sum_{k=1}^{K} \mathbf{G}_k \cdot (\mathbf{u}_k - \mathbf{u}_k^*), \quad (4.1)$$

for the fields of Lagrange multipliers, $\mathbf{N}$, $\mathbf{M}$ pertaining to (i) and $\mathbf{P}$, M, $\mathbf{G}_k$ pertaining to (v), must belong to the conjugate spaces of forces. Direct examination of the variation of I_1 affirms this statement. Following the same pattern that led from (2.7) to (2.10), with particular emphasis on the fact that the geometry of $\mathscr{C}_u$ is already determined, we can derive δI_1 in the form

$$\delta I_1 = \iint_{\mathscr{M}} \left\{ \left(\frac{\delta \Sigma}{\delta \gamma} - \mathbf{N} \right) \cdot \delta \gamma + \left(\frac{\delta \Sigma}{\delta \chi} - \mathbf{M} \right) \cdot \delta \chi \right.$$

$$\left. - [\gamma - \gamma(\mathbf{u})] \cdot \delta \mathbf{N} - [\chi - \chi(\mathbf{u})] \cdot \delta \mathbf{M} - [\mathbf{T}\nabla + \mathbf{p}] \cdot \delta \mathbf{u} \right\} dA$$

$$+ \int_{\mathscr{C}_f} \{[\mathbf{T}\nu - \mathbf{B} + [(\mathbf{F} + \mathbf{t} \times \mathbf{k})\mathbf{R}']_{,s}] \cdot \delta \mathbf{u}$$

$$+ (J\bar{\nu}\Gamma \mathbf{M}\nu + \mathbf{k} \cdot \mathbf{t})\mathbf{t} \cdot \delta \omega\} \, ds - \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{t}_i \times \mathbf{k}_i)\mathbf{R}^i \cdot \delta \mathbf{u}_i$$

$$- \int_{\mathscr{C}_u} [(\mathbf{u} - \mathbf{u}^*) \cdot \delta \mathbf{P} + \mathbf{n} \cdot \mathbf{t}^* \times \mathbf{n}^* \delta M] \, ds - \sum_{k=1}^{K} (\mathbf{u}_k - \mathbf{u}_k^*) \cdot \delta \mathbf{g}_k$$

$$+ \int_{\mathscr{C}_u} \{[\mathbf{T}^*\nu + (\mathbf{F}^*\mathbf{R}^{*\prime})_{,s} - \mathbf{P}] \cdot \delta \mathbf{u} + (\bar{\nu}^*\Gamma \mathbf{M}\nu - M)\mathbf{t} \cdot \delta \omega\} \, ds$$

$$- \sum_{k=1}^{K} (\mathbf{F}_k^* \mathbf{R}^{*\prime} + \mathbf{G}_k) \cdot \delta \mathbf{u}_k, \quad (4.2)$$

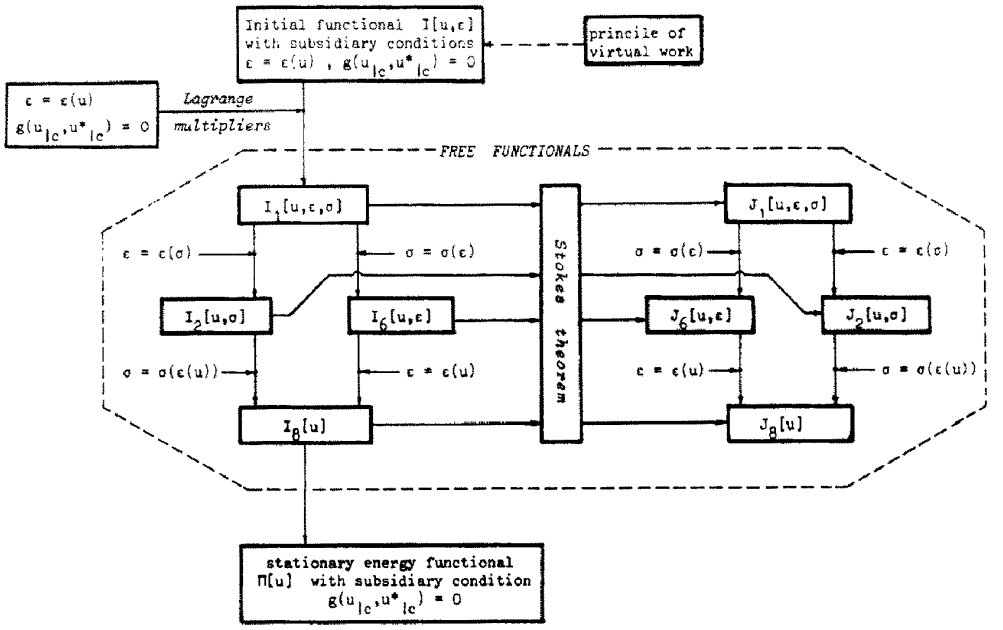

Fig. 1.

where $\mathbf{F}^* = \mathbf{F}(\mathbf{u}^*, \sigma_{|c})$ and $\mathbf{T}^* = \mathbf{T}(\mathbf{u}^*, \sigma_{|c})$. The set of the stationarity conditions for the functional I_1 contains not only the full system (i)–(v), but also includes relations identifying the Lagrange multipliers $\mathbf{P}$, M, $\mathbf{G}_k$, as the effective boundary force and bending moment appearing along $\mathscr{C}_u$ due to the imposition of the geometric constraints (v) there. These relations placed in the last lines of (4.2) can serve to eliminate the multipliers $\mathbf{P}$, M, $\mathbf{G}_k$ from I_1 and so to unify its structure. Their substitution to (4.1) results in the following change of the boundary part I_b of the functional I_1:

$$I_b[u_{|c}, \sigma_{|c}] = \int_{\mathscr{C}_f} [\Phi + \Psi]\, ds + \int_{\mathscr{C}_u} \{[\mathbf{T}^*\nu + (\mathbf{F}^*\mathbf{R}^{*\prime})_{,s}] \cdot (\mathbf{u} - \mathbf{u}^*) + (J\bar{\nu}^*\boldsymbol{\Gamma}\mathbf{M}\nu)\bar{\mathbf{n}} \cdot \bar{\nu}^*\}\, ds - \sum_{k=1}^{K} \mathbf{F}_k^*\mathbf{R}^{*\prime} \cdot (\mathbf{u}_k - \mathbf{u}_k^*). \quad (4.3)$$

Similar substitution to the expression for the variation (4.2) results in vanishing of the last lines, leaving, in consequence, only the relations (i)–(v) as the stationarity conditions of I_1.

Now the constraints included in the initial problem (3.14) may be removed and the problem itself can be restated as:

$$\text{find } [u_0, \varepsilon_0, \sigma_0] \in \mathscr{S} \text{ such that } \delta I_1[u_0, \varepsilon_0, \sigma_0] = 0. \quad (4.4)$$

This is the Hu–Washizu variational principle.

The functional I_1 comprises explicitly the relations of the geometric type (i) and (v). The question arises whether it is possible to effect any transformation that would promote explicit occurrence of the static relations (ii) and (iii). Let us point out that the following equation holds true:

$$\mathbf{N} \cdot \boldsymbol{\gamma}(\mathbf{u}) + \mathbf{M} \cdot \boldsymbol{\chi}(\mathbf{u}) = [\mathbf{u}\mathbf{T} + \mathbf{u}\mathbf{Q} - \mathbf{n}\boldsymbol{\Gamma}\mathbf{M}\nu] \cdot \nabla - \mathbf{u} \cdot \mathbf{T}\nabla - \mathbf{u} \cdot \mathbf{Q}\nabla, \quad (4.5)$$

where

$$\mathbf{Q} = \tfrac{1}{2}(\mathbf{u} \otimes \nabla)[\mathbf{N} + (\mathbf{M} \cdot \mathbf{b})\mathbf{a}] - [(\mathbf{m} - \mathbf{n}) \otimes \nabla]\mathbf{M} + \tfrac{1}{2}[(\boldsymbol{\Gamma}\mathbf{M})\nabla] \times (\mathbf{u} \otimes \nabla)\boldsymbol{\varepsilon}.$$

Substituting this result to (4.1) and applying the Stokes theorem to the terms under the

divergence operator [ ]∇ we can derive an equivalent form of the functional I_1:

$$J_1[u,\varepsilon,\sigma] = -\iint_{\mathscr{A}} [(\mathbf{N}\cdot\boldsymbol{\gamma}+\mathbf{M}\cdot\boldsymbol{\chi}-\Sigma)+(\mathbf{T}\nabla+\mathbf{p})\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{Q}\nabla+(\Pi-\mathbf{p}\cdot\mathbf{u})]\,dA+J_b[u_{|c},\sigma_{|c}], \tag{4.6}$$

where the boundary part J_b is

$$\begin{aligned} J_b[u_{|c},\sigma_{|c}] = \int_{\mathscr{C}_f} \{&[\mathbf{T}\nu-\mathbf{B}+(\mathbf{FR}')_{,s}]\cdot\mathbf{u}-(\Phi-\mathbf{B}\cdot\mathbf{u}) \\ &+(\mathbf{Q}\nu)\cdot\mathbf{u}-\mathbf{FR}'\cdot\mathbf{t}-\mathbf{n}\boldsymbol{\Gamma}\mathbf{M}\nu-\Psi\}\,ds-\sum_{i=1}^{N}\mathbf{F}_i\mathbf{R}'\cdot\mathbf{u}_i \\ &+\int_{\mathscr{C}_u}\{[\mathbf{T}^*\nu+(\mathbf{F}^*\mathbf{R}^{*\prime})_{,s}]\cdot\mathbf{u}+(\mathbf{Q}^*\nu)\cdot\mathbf{u}-\mathbf{F}^*\mathbf{R}^{*\prime}\cdot\mathbf{t} \\ &-\mathbf{n}\boldsymbol{\Gamma}\mathbf{M}\nu-(\mathscr{F}\tilde{\nu}^*\boldsymbol{\Gamma}\mathbf{M}\nu)\tilde{\mathbf{n}}\cdot\tilde{\nu}^*\}\,dS+\sum_{k=1}^{K}\mathbf{F}_k^*\mathbf{r}^{*\prime}\cdot\mathbf{u}_k. \end{aligned} \tag{4.7}$$

Let us note some specific features of the functional J_1. Owing to the equivalence between I_1 and J_1 the variations of both functionals are equal, $\delta I_1 = \delta J_1$, and hence their stationarity conditions are the same. The first three terms under the surface integral in (4.6) are equal in value to the complementary energy function Σ^c (3.9). Besides, expressions equivalent to the complementary work Π^c and Φ^c (3.12) have appeared under the surface and the $\mathscr{C}_f$-boundary integrals. Due to strong interference of rotations and arising hence nonlinearities the equilibrium equations (iii) could be separated in the surface integrand at the cost of including a new term $\mathbf{Q}$, having the character of residual internal forces. Due to the same reason it seems virtually impossible to handle the boundary integrands in a similar way.

A functional of this type, termed the total complementary energy functional, was brought out in [9, 13–16]. Thus the variational principle $\delta J_1 = 0$ may be called the principle of generalized complementary energy.

5. DERIVED VARIATIONAL PRINCIPLES

In the original boundary-value problem (i)–(v) the fields ε and σ can be eliminated via substitution of the kinematic relations (i) and the constitutive equations (iv) to the static relations (ii) and (iii), thus gradually reducing the space of states $\mathscr{S}$. This process has its reflection in the variational approach.

Both functionals derived in the previous section, I_1 and J_1, have the same stationarity conditions that result from (4.2). Adding them to these functionals in the form of supplementary constraints and then eliminating with their help some of the variable fields does not affect the stationarity conditions of the resulting reduced functionals. We shall apply this procedure to obtain six subsequent free variational principles.

Let us eliminate the strains ε from the functionals I_1 and J_1 by means of the inverted constitutive equations $\varepsilon = \varepsilon(\sigma)$ (3.9). As a result we obtain two free functionals

$$I_2[u,\sigma] = \iint_{\mathscr{A}} (-\Sigma^c(\sigma)-\Pi+\mathbf{N}\cdot\boldsymbol{\gamma}(\mathbf{u})+\mathbf{M}\cdot\boldsymbol{\chi}(\mathbf{u})]\,dA-I_b[u_{|c},\sigma_{|c}], \tag{5.1}$$

$$J_2[u,\sigma] = -\iint_{\mathscr{A}} [\Sigma^c(\sigma)+(\mathbf{T}\nabla+\mathbf{p})\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{Q}\nabla+(\Pi-\mathbf{p}\cdot\mathbf{u})]\,dA+J_b[u_{|c},\sigma_{|c}]. \tag{5.2}$$

Note that the equality $\delta I_2 = \delta J_2$ is still preserved. To obtain the expression for δI_2 one should replace the first two lines in (4.2) under the surface integral, i.e. the ones containing the constitutive equations and the kinematic relations, with the relations of the form $[\varepsilon(\sigma) - \varepsilon(u)]\,\delta\sigma$.

The sense of the operation just completed consists in reducing the space $\mathscr{S}$ to the space $\mathscr{S}_R$ of the pairs $[u, \sigma]$ so that we may replace the problem (4.3) by a modified unconstrained problem:

$$\text{find } [u_0, \sigma_0] \in \mathscr{S}_R \text{ such that } \delta I_2[u_0, \sigma_0] = 0 \text{ or } \delta J_2[u_0, \sigma_0] = 0. \tag{5.3}$$

This is the Hellinger–Reissner variational principle.

A similar operation, based on the elimination of the stress field σ via the constitutive equations in their direct form $\sigma = \sigma(\varepsilon)$ (3.5), leads to another pair of free functionals:

$$I_6[u, \varepsilon] = -\iint_{\mathscr{M}} \left\{\Sigma(\varepsilon) + \Pi - h\mathbf{H}\cdot\left[\gamma \otimes \gamma(\mathbf{u}) + \frac{h^2}{12}\chi \otimes \chi(\mathbf{u})\right]\right\} \mathrm{d}A - I_b[u_{|c}, \sigma(\varepsilon_{|c})], \tag{5.4}$$

$$J_6[u, \varepsilon] = -\iint_{\mathscr{M}} \{\Sigma(\varepsilon) + [\mathbf{T}(\mathbf{u}, \sigma(\varepsilon))\nabla + \mathbf{p}]\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{Q}(u, \sigma(\varepsilon))\nabla + (\Pi - \mathbf{p}\cdot\mathbf{u})]\}\,\mathrm{d}A + J_b[u_{|c}, \sigma(\varepsilon_{|c})]. \tag{5.5}$$

In the derivation of J_6 use has been made of the relation

$$2\Sigma(\varepsilon) = \mathbf{N}(\gamma)\cdot\gamma + \mathbf{M}(\chi)\cdot\chi. \tag{5.6}$$

Again the equality $\delta I_6 = \delta J_6$ holds true and to obtain the variation of these functionals the first line in (4.2), now vanishing identically, should be removed, and elsewhere the right-hand sides of (3.5) should be substituted for $\mathbf{N}$ and $\mathbf{M}$. The functionals I_6 and J_6 are accompanied with a variational principle similar to (5.3).

Carrying this process on we can eventually derive a pair of free functionals depending solely on the displacement field u. These two functionals result either from I_6 and J_6 via substitution of the kinematic relations $\varepsilon = \varepsilon(u)$, or from I_2 and J_2 via substitution of the compound relation $\sigma = \sigma(\varepsilon(u))$ obtained by introduction of the kinematic relations to the constitutive equations. Both of the two ways lead to the functionals

$$I_8[u] = \iint_{\mathscr{M}} [\Sigma(\varepsilon(\mathbf{u})) - \Pi]\,\mathrm{d}A - I_b[u_{|c}, \sigma(\varepsilon(u_{|c}))], \tag{5.7}$$

$$J_8[u] = -\iint_{\mathscr{M}} \{\tfrac{1}{2}[\mathbf{T}(u, \sigma(\varepsilon(u)))\nabla + \mathbf{p}]\cdot\mathbf{u} + \tfrac{1}{2}\mathbf{Q}(u, \sigma(\varepsilon(u)))\nabla\cdot\mathbf{u} + (\Pi - \mathbf{p}\cdot\mathbf{u})\}\,\mathrm{d}A + \tfrac{1}{2}J_b[u_{|c}, \sigma(\varepsilon(u_{|c}))] - \tfrac{1}{2}\int_{\mathscr{K}_f} [(\Phi - \mathbf{B}\cdot\mathbf{u}) + \Psi]\,\mathrm{d}s - \tfrac{1}{2}\int_{\mathscr{K}_u} (J\bar{\mathbf{v}}^*\boldsymbol{\Gamma}\mathbf{M}\nu)\bar{\mathbf{n}}\cdot\bar{\mathbf{v}}^*\,\mathrm{d}s, \tag{5.8}$$

where the relations (5.6) and (4.5) have been employed additionally in the derivation of J_8. Their stationarity conditions are the static relations (ii) and (iii), expressed in displacements, and the geometric boundary conditions (v).

The solution to the problem

$$\text{find } [u_0] \in \mathscr{U} \text{ such that } \delta I_8[u_0] = 0 \text{ or } \delta J_8[u_0] = 0 \tag{5.9}$$

is the actual displacement field of the deformed surface $\bar{\mathscr{M}}$.

Finally, reversing the procedure underlying the derivation of the Hu–Washizu functional I_1 from the initial functional I, we can exclude the geometric boundary conditions (v) from I_8 to obtain the functional of the total potential energy

$$\Pi[u] = \iint_{\mathcal{M}} [\Sigma(\varepsilon(u)) - \Pi]\, dA - \int_{\mathcal{C}_f} (\Phi + \Psi)\, ds. \tag{5.10}$$

In fact, $\Pi[u]$ could have been obtained directly from $I[u, \varepsilon]$ (3.13) via substitution of the kinematic relations $\varepsilon = \varepsilon(u)$ there. Therefore the variation $\delta\Pi$ coincides with the left-hand side of (2.10) after substitution of the relations $\sigma = \sigma(\varepsilon(u))$ for **N** and **M**. Consequently, the stationarity conditions of Π are the static relations (ii) and (iii) expressed in displacements. The associated variational problem can be stated as

$$\text{find } [u_0] \in \mathscr{U} \text{ such that } \Pi[u_0] = \underset{u_{|c} \in \mathscr{D}_u}{\text{stationary values of}}\, \Pi[u] \tag{5.11}$$

where $\mathscr{D}_u = \{u_{|c} \mid g(u_{|c}, u^*) = 0\}$. This is the stationary potential energy principle.

Closing this section let us remark that the elimination of the displacement field **u** expressed in the strains ε from the Hellinger–Reissner functionals I_2 and J_2 would lead to the complementary functionals and variational principles. However, the problem of inverting the nonlinear kinematic relations $\varepsilon = \varepsilon(u)$ is nontrivial and has not yet been solved in the general case. The existing solution, due to Washizu[39], reaches only as far as the Donnel–Marguerre nonlinear theory of shallow shells. Clearly, the obstacle is of a geometric nature, stemming from the fact that more than one displacement state u may correspond to a given surface strain γ and from the coupling between the surface strain and change of curvature measures through the compatibility equations. It seems that the solution should be sought in modification of the attitude towards the deformation measures and the approach developed by de Veubeke provides with some sort of alternative here (see [21–23]).

6. FINAL REMARKS

The functionals and variational principles presented in the main body of this paper deserve some additional comments. First of all, they are grouped in two equivalent families exhibiting the properties of symmetry (cf. Fig. 1). The pairs $I_k - J_k$ are related by the Stokes theorem. Their variations are always equal to each other: $\delta I_k = \delta J_k$. The functionals of the $\mathscr{I}$ family contain the geometric relations of the theory in the explicit form, whereas those of the $\mathscr{J}$ family the static relations. Consequently, either of them lays down a suitable basis for the problems with some additional constraints of the geometric or static character, respectively (cf. [9]).

Secondly, the question concerning the character of the stationary points should be mentioned as still an open one. So far the conditions for attainment of infima are thoroughly examined only in the theory of convex functionals[40] and the related theory of potential monotone operators[26, 32, 33]. Yet the variational problems arising in the general geometrically nonlinear theory of shells may turn out to be nonconvex due to the interference of the nonlinear strain–displacement relations or nonmonotonicity of the external loads. Hence, nonuniqueness and branching of solutions should be expected. The clarification of this question would be of essential importance for the problems of elastic stability.

Finally, if the external loads involved in the problem are nonpotential the incremental formulations can be employed. The variational principles presented herein should prove helpful even in this case. For these and related problems see [28].

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